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A COMPUTATIONAL APPROACH OF TURBULENT MIXING. THE TRAJECTORIES ANALYSIS

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Abstract After hundreds of years of stability study, the problems of flow kinematics are far from complete solving. A modern theory appears in this field: the mixing theory. Its mathematical methods and techniques developed the significant relation between turbulence and chaos. The turbulence is an important feature of dynamic systems with few freedom degrees, the so-called "far from equilibrium systems". These are widespread between the models of excitable media.

In [2] it was studied the efficiency of mixing for a periodic 2Dmodel, as function of time, from discrete standpoint. A useful conclusion was that more irrational values for the versors would give interesting properties in discrete time, comparing to 3D case [1]. On the other hand, the graphic analysis realized in [3] involved that in 2D case, the mixing has also a nonlinear behavior and the rare events can appear.

In this paper a computational analysis for 2D modified mixing model is started. Namely, the trajectories form is analyzed for different values of the parameters of the Cauchy-Green tensor. For the simulations there are used specific procedures and functions of Maple VI. The simulations are used for further comparisons with basic 2D model behavior, and also for experimental matches [6].

Key words: mixing, flow, Cauchy-Green tensor

1. Introduction

The turbulence mathematical term can be defined as "chaotic behavior of far from equilibrium systems, with very few freedom degrees". In this area two important theories are distinguished:

a) The transition theory from smooth laminar flows to chaotic flows, characteristic to turbulence.

b) Statistic studies of the complete turbulent systems.

The statistical idea of flow is represented by the map:

$$x=\Phi_t(X), X=\Phi_{t=0}(X) \qquad (1)$$

In the continuum mechanics the relation (1) is named *flow*, and it is a diffeomorphism of class C^k. Moreover, (1) must satisfy the relation:

$$0\langle J\langle\infty, J=\det\left(\frac{\partial x_i}{\partial X_j}\right), \qquad (2)$$
$$J=\det(D(\Phi_t(X)))$$

where D denotes the derivation with respect to the reference configuration, in this case **X**. The relation (2) implies two particles, X_1 and X_2, which occupy the same position **x** at a moment. Non-topological behavior (like break up, for example) *is not allowed*.

With respect to **X** there is defined the basic measure of deformation, the *deformation gradient*, **F,** namely:

$$\mathrm{F}=(\nabla_X \Phi_t(\mathrm{X}))^T,\ F_{ij}=\left(\frac{\partial x_i}{\partial X_j}\right) \qquad (3)$$

where ∇_X denotes differentiation with respect to **X**. According to (3), **F** is non singular. The basic measure for the deformation with respect to **x** is the *velocity gradient* [5].

After defining the basic deformation of a material filament and the corresponding relation for the area of an infinitesimal material surface [1, 3], we can define the basic deformation measures: the *length deformation* λ and *surface deformation* η, with the relations [1, 5]:

$$\lambda=(\mathrm{C}:\mathrm{MM})^{1/2},\eta=(\det F)\cdot\left(\mathrm{C}^{-1}:\mathrm{NN}\right)^{1/2} \qquad (4)$$

with **C** (=$\mathbf{F}^{\mathrm{T}}\cdot\mathbf{F}$) the *Cauchy-Green deformation tensor*, and the vectors **M**, **N** are the orientation versors in length and surface respectively. The scalar form for (4), used in practice, is:

$$\lambda^2 = C_{ij} M_i N_j, \; \eta^2 = (\det F) \cdot C_{ij}^{-1} M_i N_j \qquad (5)$$

with $\sum M_i^2 = 1$, $\sum N_j^2 = 1$, the condition for the versors.

The deformation tensor **F** and the associated tensors **C**, $\mathbf{C}^{-1}$ represents the basic quantities in the deformation analysis for the infinitesimal elements.

Studying a mixing for a flow implies the analysis of successive stretching and folding phenomena for its particles, the influence of parameters and initial conditions [5]. In the previous works, the study of the 3D non-periodic models exhibited a quite complicated behavior [1]. In agreement with experiments, they involved some significant events - the so-called "rare events". The variation of parameters had a great influence on the length and surface deformations. The 2D case was simpler, but significant events can issue for irrational values of the length and surface versors, as was the situation in 3D case.

In this paper, a little modification of the basic 2D model [1, 5] is taken into account. For this modified model, the Cauchy-Green tensor is calculated, and its variation for different parameter values is studied.

2. The modified 2D mixing model

Let us consider the modified 2D mixing model:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 + x_1 \\ \dot{x}_2 = K \cdot G \cdot x_1 + x_2 \end{cases} \qquad (6)$$

with $-1 < K < 1,\; 0 < G < 1$

Here is the time derivative. If we attach the initial condition:

$$x_1(0) = X_1, x_2(0) = X_2 \qquad (7)$$

The Cauchy problem (6)÷(7) has the following calculated solution:

$$x_1 = \left[\frac{X_2}{2} \cdot \frac{2+KG^2-\sqrt{2+KG^2}}{KG} - \frac{X_1}{2} \cdot \left(\frac{2+KG^2}{\sqrt{2+KG^2}} + 1 \right) \right] \cdot \exp\left(1-\sqrt{2+KG^2}\right)t + \qquad (8)$$

$$\left[\frac{X_1}{2} \cdot \left(\frac{2+KG^2}{\sqrt{2+KG^2}} + 1 \right) - \frac{X_2}{2} \cdot \frac{\left(1+KG^2\right) \cdot \sqrt{2+KG^2}}{KG} \right] \cdot \exp\left(1+\sqrt{2+KG^2}\right)t$$

$$x_2 = \left(\frac{X_2}{2} - \frac{X_1}{2} \cdot \frac{KG}{\sqrt{2+KG^2}} \right) \cdot \exp\left(1-\sqrt{2+KG^2}\right)t + \left[\frac{X_1}{2} \cdot \frac{KG}{\sqrt{2+KG^2}} + \frac{X_2}{2} \cdot \left(1-\sqrt{2+KG^2}\right) \right] \cdot \exp\left(1+\sqrt{2+KG^2}\right)t$$

The deformation gradient (3) is found as:

$$F = \begin{pmatrix} -\frac{1}{2} \cdot \left(\frac{2+KG^2}{\sqrt{2+KG^2}} - 1 \right) \cdot \exp\left(1-\sqrt{2+KG^2}\right)t + \frac{1}{2}\left(\frac{2+KG^2}{\sqrt{2+KG^2}} + 1 \right) \cdot \exp\left(1+\sqrt{2+KG^2}\right)t & \frac{1}{2} \cdot \frac{2+KG^2-\sqrt{2+KG^2}}{KG} \cdot \exp\left(1-\sqrt{2+KG^2}\right)t - \frac{1}{2} \cdot \frac{\left(1+KG^2\right)\sqrt{2+KG^2}}{KG} \cdot \exp\left(1+\sqrt{2+KG^2}\right)t \\ -\frac{1}{2} \cdot \frac{KG}{\sqrt{2+KG^2}} \cdot \exp\left(1-\sqrt{2+KG^2}\right)t + \frac{1}{2} \cdot \frac{KG}{\sqrt{2+KG^2}} \cdot \exp\left(1+\sqrt{2+KG^2}\right)t & \frac{1}{2} \cdot \exp\left(1-\sqrt{2+KG^2}\right)t + \frac{1}{2}\left(1-\sqrt{2+KG^2}\right) \cdot \exp\left(1+\sqrt{2+KG^2}\right)t \end{pmatrix} \qquad (9)$$

The transposed matrix $\mathbf{F}^{\mathbf{T}}$ follows immediately, also with two exponentials on each position, and thus, the Cauchy-Green tensor $C = F^T \cdot F$ is in our case, the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \qquad (10)$$

With the notation

$$KG^2 = \gamma \qquad (11)$$

and taking into account that $\gamma \leq 1$, the calculated components are:

$$c_{11}=\frac{\gamma^2\cdot(1+K)+(5-2K)\cdot\gamma+6}{4\gamma K}\cdot\exp\left(\left(1-\sqrt{2+\gamma}\right)t\right)+\frac{\gamma^3+(K+2)\gamma^2+(1+7K)\gamma+2}{4\gamma K}\cdot\exp\left(\left(1+\sqrt{2+\gamma}\right)t\right)-\frac{(K+1)\gamma^2+(K-3)\gamma+6K+4}{2\gamma K}\cdot\exp(2t) \qquad (12.1)$$

$$c_{12}=\frac{1}{4}\cdot\left[\frac{\left(-\gamma^2-\gamma-2\right)\cdot\sqrt{2+\gamma}-4\gamma-4}{\gamma(2+\gamma)}\right]\cdot\exp\left(\left(1-\sqrt{2+\gamma}\right)t\right)+\frac{1}{4}\cdot\left[\frac{(3\gamma+6)\cdot\sqrt{2+\gamma}-2\gamma^3-2\gamma^2+3\gamma+2}{\gamma(2+\gamma)}\right]\cdot\exp((-1-\gamma)t)+\frac{1}{4}\cdot\left[\frac{\left(-2\gamma^2-3\gamma-2\right)\cdot\sqrt{2+\gamma}+2\gamma^3+6\gamma^2+3\gamma-2}{\gamma(2+\gamma)}\right]\cdot\exp\left(\left(1+\sqrt{2+\gamma}\right)t\right) \qquad (12.2)$$

$$c_{21}=\frac{1}{4}\cdot\frac{\gamma\left(2+\gamma-\sqrt{2+\gamma}\right)}{2+\gamma}\cdot\exp\left(\left(1-\sqrt{2+\gamma}\right)t\right)+\frac{1}{4}\cdot\frac{\gamma\left(2+\gamma+\sqrt{2+\gamma}\right)}{2+\gamma}\cdot\exp\left(\left(1+\sqrt{2+\gamma}\right)t\right)-\frac{1}{4}\cdot 2\gamma\cdot\exp((-1-\gamma)t) \qquad (12.3)$$

$$c_{22}=\frac{1}{4}\cdot\left[\frac{\gamma^2}{2+\gamma}+1\right]\cdot\exp\left(\left(1-\sqrt{2+\gamma}\right)t\right)+\frac{1}{4}\cdot\left[\frac{\gamma^2}{2+\gamma}+\left(1+\sqrt{2+\gamma}\right)^2\right]\cdot\exp\left(\left(1+\sqrt{2+\gamma}\right)t\right)+\frac{1}{4}\cdot\left[\frac{-2\gamma^2}{2+\gamma}+2\left(1-\sqrt{2+\gamma}\right)\right]\cdot\exp((-1-\gamma)t) \qquad (12.4)$$

3. Results

3.1. Graphic analysis

In what follows we neglect the factor ¼. Three different -random- values where chosen for the parameter γ:

3.1. $\gamma = 0.08$;
3.2. $\gamma = 0.25$;
3.3. $\gamma = 0.85$.

For each case, the Cauchy-Green tensor was evaluated, from the trajectories analysis standpoint. The Maple VI software was used, for plotting the trajectories in discrete time. The plots were grouped as follows: c_{11} + c_{12} in a coordinate system, and c_{21} + c_{22} in another coordinate system. For the first plot the color is red, for the second, blue. The plots are as follows:

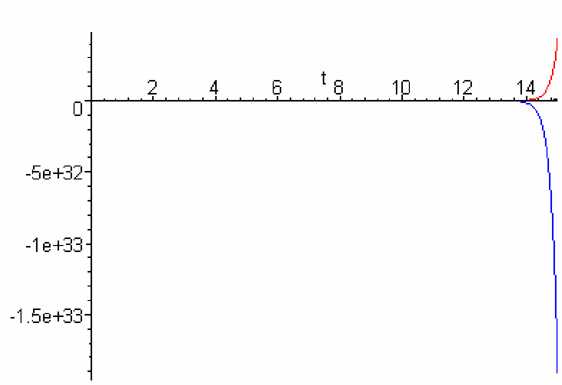

Figure 3.1.1. c_{11} and c_{12}

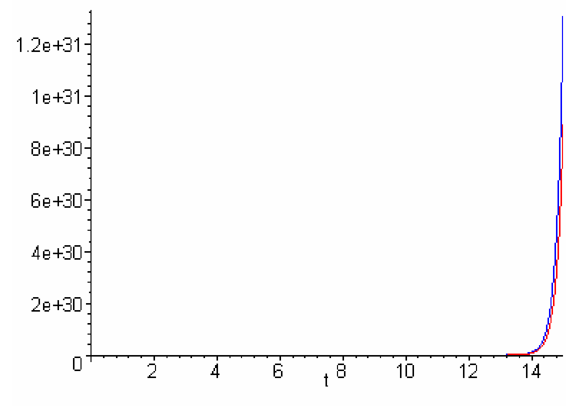

Figure 3.1.2. c_{21} and c_{22}

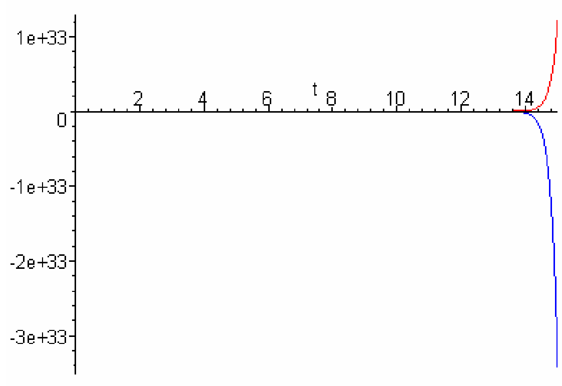

Figure 3.2.1. c_{11} and c_{12}

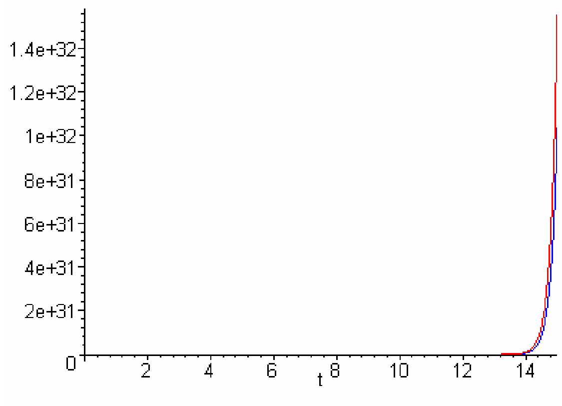

Figure 3.2.2. c_{21} and c_{22}

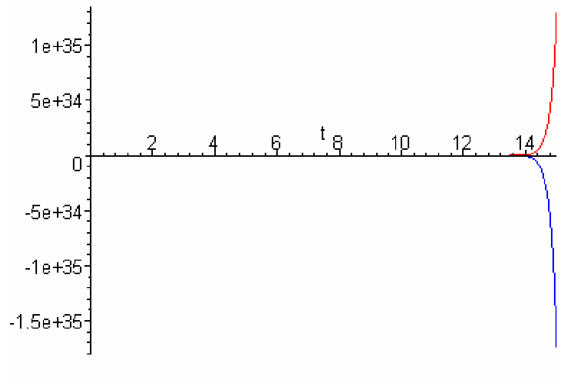

Figure 3.3.1. c_{11} and c_{12}

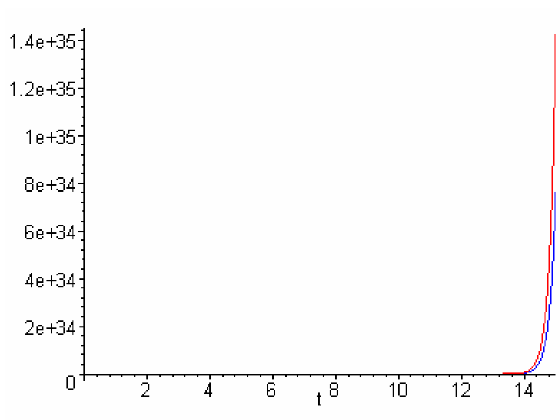

Figure 3.3.2. c_{21} and c_{22}

3.2. Remarks

Some remarks and further aims must be noted:

- it has been taken only positive values of γ, for moment; negative values would produce some symmetric plots;
- the time scale were chosen 0 ... 15, since otherwise a *breakout* of simulation were noticed;
- thus, it can be assessed that the plots are *not bounded*, although continues, as function of time;
- the Cauchy-Green deformation tensor *can produce rare events*, for a larger simulation time;
- next target is to calculate the inverse, $\mathbf{C}^{-1}$ of the deformation tensor, and to study the deformations in length and surface [5], from analytical and numeric standpoint; thus a comparison with the basic model would produce useful features.

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