# DYNAMICAL SYNTHESIS OF OSCILLATING SYSTEM MASS-SPRING-MAGNET

**Rumen NIKOLOV, Todor TODOROV**
Technical University of Sofia, Bulgaria

**Abstract.** On the basis of a qualitative analysis of nonlinear differential equation, which describes the behavior of the oscillating system mass-spring-magnet, a dynamical synthesis is performed. The magnetic force is obtained using the finite element method and approximated by an analytical expression. The parameters of the system which ensure periodical oscillations with given properties are determined.

**Keywords:** nonlinear dynamics, qualitative analysis, phase portrait, separatrix, magnetic force, FEM

## 1. Introduction

The tasks of dynamical synthesis of mechanical systems come down to determining of mass and force parameters depending on preliminarily described manner of behavior. For example, in the Theory of Mechanisms and Machines a typical problem of such a kind is the finding of the moment of inertia of a flywheel, which ensures a given irregularity of motion [1]. Tasks of the same type concern the time-response or the durability of transient states [2, 3].

Magneto-mechanical systems are with sophisticated characteristics and the description of their behavior is based on solving of nonlinear ODEs [4, 5], and in the general case it relates to investigation of complicated dynamical processes which are described by PDEs [6]. The choice of the parameters of such a system also can be added to the tasks of the dynamical synthesis, as far as such a choice can guarantee some requested properties of the real system.

At the present paper a nonlinear oscillating system of the type mass-spring-magnet is investigated. On the basis of a qualitative analysis a method for achieving of given parameters of the oscillations is proposed.

## 2. Magnetic force determination

The basic elements of the system are shown in Figure 1. Ferromagnetic mass is fixed to the free end of cantilever and oscillates in the force field of a permanent magnet. When the cantilever is in undistorted shape and it is in horizontal position between the faces of ferromagnetic mass and magnet, an initial air gap $h$ is established. A linear generalized coordinate $y$ is chosen for describing of the mass oscillations. Zero position of this coordinate corresponds to the horizontal position of the cantilever.

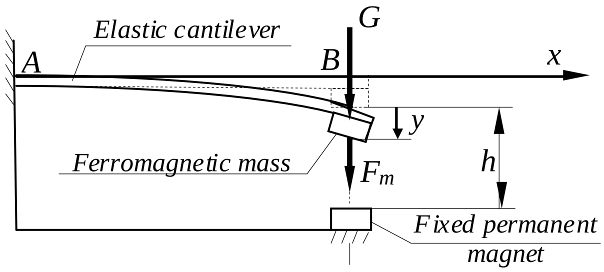


Figure 1. Scheme of the oscillating system

In order the oscillating system to be qualitatively investigated, it is necessary to be determined the force dependence on the position. This function is worked out for any particular case and it depends on dimensions, shape and material properties.

For more precise determination of the magnetic force it is made an investigation of the magnetic field by FEM. The system that is investigated is shown in Figure 2. It consists of a cylindrical permanent magnet 1 and a ferromagnetic cylinder 2, disposed over the magnet. All dimensions are denoted in Figure 2.

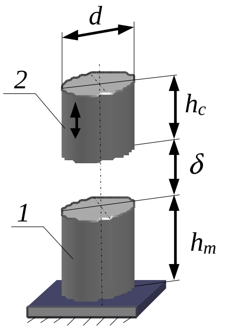


Figure 2. System for determining of magnetic force
*1* – permanent magnet; *2* – ferromagnetic cylinder

The magnetic field is analyzed as an axe-symmetrical one. The task of analysis is formulated by Poisson's equation with respect to the magnetic vector-potential in the cylindrical coordinate system. The investigated area is determined as sufficient large buffer zone (of the order of 20-multiple dimension of the system) at the boundary of which they are imposed homogenous boundary conditions of Dirichlet.

The problem is solved with the help of the software product FEMM (Finite Element Method Magnetics), and for automation of the calculations some programs in the Lua Script® language are created. For determining of the magnetic force the built in tension tensor of Maxwell is used. The number of mesh nodes of the finite elements varies for the different tasks between 18,000 and 20,000.

Investigations are made for two types of permanent magnets (BaFerrite and NDFeB) and two dimensions of the diameters (8 and 13 mm). Results for the magnetic field distribution and for the attractive force between magnet and ferromagnetic cylinder are obtained when the air gap changes from 1 to 5 mm.

Magnetic field lines for diameter 8 mm and permanent BaFerrite for air gaps 1 and 5 mm are visualized in Figures 3a and 3b. For the same dimensions an investigation is made for the permanent magnet NdFeB. Results for the magnetic field distribution are given in Figure 3c.

As a consequence, there is no a principle difference in the distribution field patterns of both types of magnets. However the difference in terms of force is significant, which can be observed in Table 1. For the magnet of NdFeB the mean magnetic force is more of 60 times bigger than the same force in the system of BaFerrite.

Table 1. Force variation

| $\delta$ [mm] | Force [N] | | |
|---|---|---|---|
| | BaFerrite 8/15/10 | NdFeB 8/15/10 | BaFerrite 13/8/5 |
| 1 | 0.121 | 8.50 | 0.209 |
| 1.5 | 0.091 | 5.92 | 0.158 |
| 2 | 0.065 | 4.13 | 0.118 |
| 2.5 | 0.046 | 2.95 | 0.089 |
| 3 | 0.033 | 2.17 | 0.068 |
| 3.5 | 0.026 | 1.63 | 0.053 |
| 4 | 0.019 | 1.20 | 0.041 |
| 4.5 | 0.014 | 0.90 | 0.033 |
| 5 | 0.011 | 0.70 | 0.026 |

The obtained force data are approximated by the relation

$$F_m = \frac{\kappa}{(\delta_0 + \delta)^2} \tag{1}$$

which is close to Coulomb's low [7] when $\delta_0$ is small. Analogically to this law here $\kappa$ is denoted as an imaginary magnetic mass, $\delta_0$ is an imaginary initial gap. The constant values and maximal deviations are shown in Table 2. The graphs of the approximating functions are presented in Figure 4.

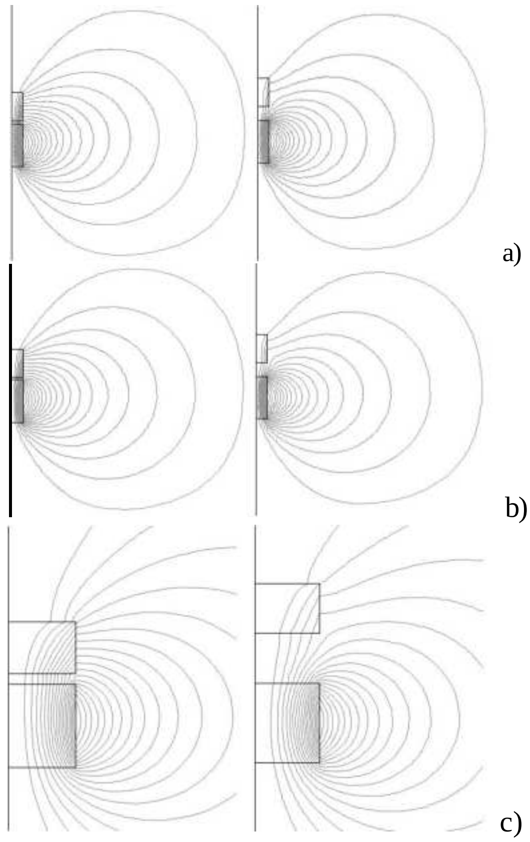


Figure 3. Magnetic field lines distribution for $\delta$ = 1 mm and $\delta$ = 5 mm
a) NdFeB $d$ = 8 mm; $h_m$ = 15 mm; $h_c$ = 10 mm;
b) BaFerrite $d$ = 8 mm; $h_m$ = 15 mm; $h_c$ = 10 mm;
c) BaFerrite $d$ = 13 mm; $h_m$ = 8 mm; $h_c$ = 5 mm

Table 2. Constant values and maximal deviations

| Material dimensions $d / h_m / h_c$ | $\kappa$ $\times10^{-7}$ [N·m$^2$] | $\delta_0$ [m] | Max. deviation $\times10^{-6}$ [N] |
|---|---|---|---|
| BaFerrite 8/15/10 | 6.2681702 | 0.0012348 | 0.00086 |
| NdFeB 8/15/10 | 35.921246 | 0.0010328 | $0.207\times10^{-6}$ |
| BaFerrite 13/8/5 | 1.4354750 | 0.0015858 | 0.000124 |

### 3. Dynamical model of the oscillating system

A mathematical model of the beam is created, taking in account the following more important simplified assumptions: all kinds of resistance are neglected; the influence of the distributions of all kind of loads is neglected, it is assumed that the mass moves rectilinearly and its rotation is ignored.

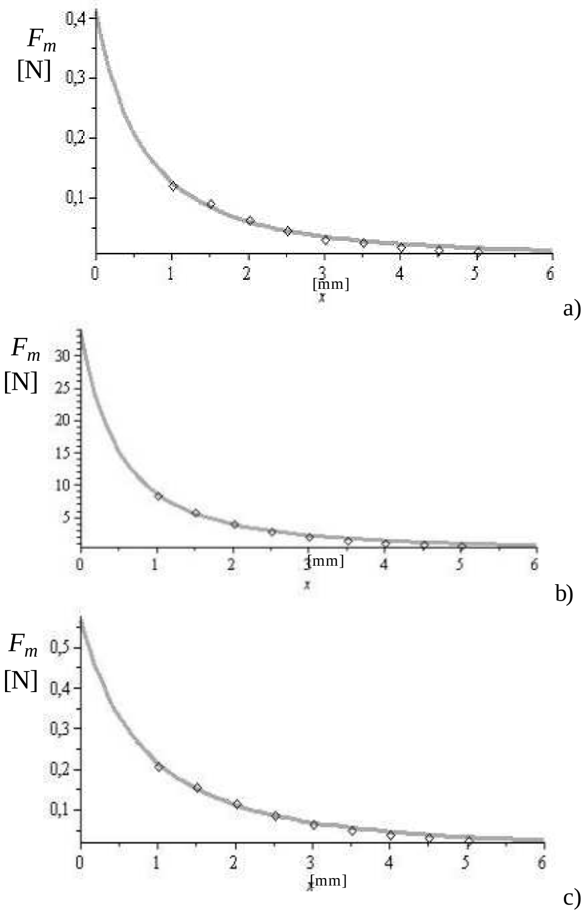


Figure 4. Approximated static characteristics of investigated cases: a) magnet BaFerrite and dimensions $d$ = 8 mm; $h_m$ = 15 mm; $h_c$ = 10 mm; b) magnet NdFeB and dimensions $d$ = 8 mm; $h_m$ = 15 mm; $h_c$ = 5 mm; c) magnet BaFerrite $d$ = 13 mm; and dimensions $h_m$ = 8 mm; $h_c$ = 5 mm

According to Figure 1 for magnetic force (1) it is yield

$$F_m = \frac{\kappa}{(\delta_0 + h - y)^2} \quad (2)$$

because $\delta = h - y$. On the mass it exerts influence the elastic force of the cantilever

$$F_e = -c \cdot y \quad (3)$$

and the weight

$$G = m \cdot g, \quad (4)$$

where $c$ is the spring stiffness, $m$ is the mass, and $g$ is the gravity acceleration.

The motion of the mass in the free cantilever end is described by the differential equation

$$m \cdot \ddot{y} + c \cdot y = m \cdot g + \frac{\kappa}{(\delta_0 + h - y)} \quad (5)$$

which after dividing to $m$ takes the view

$$\ddot{y} + k^2 \cdot y = g + \frac{\kappa}{m(\delta_0 + h - y)} \quad (6)$$

where $k$ is the natural frequency of the system. For the stroke constrains of the cantilever it follows that $y \le h$.

After substituting $h_0 = \delta_0 + h$, introducing dimensionless time $t_{new} = k^2 \cdot t$ and change of the variable

$$y = h_0 - \xi \quad (7)$$

the differential equation (6) takes the form

$$\ddot{\xi} + \xi - \alpha + \frac{\beta}{\xi^2} = 0. \quad (8)$$

Here are denoted the following parameters

$$\alpha = h_0 - \frac{g}{k^2}, \qquad \beta = \frac{\kappa}{m \cdot k^2}. \quad (9)$$

For the change of the variable it follows the constrain

$$\xi \ge \delta_0. \quad (10)$$

By the energy transformation $\ddot{\xi} = \frac{1}{2}\frac{d\dot{\xi}^2}{d\xi}$ the differential equation is yield

$$\frac{1}{2}\frac{d\dot{\xi}^2}{d\xi} = \alpha - \xi - \frac{\beta}{\xi^2} \quad (11)$$

from which after separating of the variables the first integral is worked out

$$\frac{1}{2}\dot{\xi}^2 = \alpha\xi - \frac{1}{2}\xi^2 + \frac{\beta}{\xi} + C. \quad (12)$$

By the so obtained expression after substitution with different values of the integrating constant $C$ the phase trajectories in the plane $\dot{\xi}$, $\xi$ of the oscillating system can be described [8].

## 4. Determination of the bifurcation areas

Bifurcation zones are determined by the values of the parameters $\alpha$ and $\beta$ for which the phase states of the system change its nature. For these states to be worked out the function

$$f = \alpha \cdot \xi - \frac{1}{2}\xi^2 + \frac{\beta}{\xi}, \quad (13)$$

is considered, which completely determines the phase trajectories. For this purpose Eq. (13) is presented as a sum of two functions

$$f = f_1 + f_2 \quad (14)$$

where

$$f_1 = \alpha \cdot \xi - \frac{1}{2}\xi^2, \qquad f_2 = \frac{\beta}{\xi}. \quad (15)$$

From (9), $\beta$ is a positive nonzero parameter but $\alpha$ theoretically can be also negative and zero parameter. The negative values of $\alpha$ are possible when simultaneously the air gap $h$ and the imaginary gap $\delta_0$ are small, and the natural frequency $k$ is low. Such values are relatively rarely used in the techniques. It is known that the function $f_2$ is discontinuous for $\xi = 0$ and it is strictly monotone decreasing. The function $f_1$ is a parabola with a zero root and a second root equal to $2\alpha$. They are possible two variants of the sum function $f$. The first one is it to be a strictly monotone decreasing function, and the second one is it to possess two extremes. At the first case there will not be a singular point in the phase trajectories. At the second case two singular points will occur, that correspond to the stable and unstable equilibrium points. The limit case between the two states is a point of inflection in the function $f$. For this it follows the appearance of a cusp point in the phase trajectories. From this limit state the condition of bifurcation is obtained, which is expressed as simultaneously null of first and second derivation of $f$. It follows to the system

$$\alpha - \xi - \frac{\beta}{\xi^2} = 0\,; \qquad \frac{2\beta}{\xi^3} - 1 = 0\,. \tag{16}$$

From the system (16) it is obtained

$$\beta = \frac{1}{2}\xi^3; \quad \alpha = \frac{3}{2}\xi, \tag{17}$$

which is a parametric equation of the bifurcation curve. After isolating of the parameter $\xi$ the equation of the bifurcation curve

$$\beta = \frac{4}{27}\alpha^3 \tag{18}$$

is obtained. The graph of this curve is presented in Figure 5.

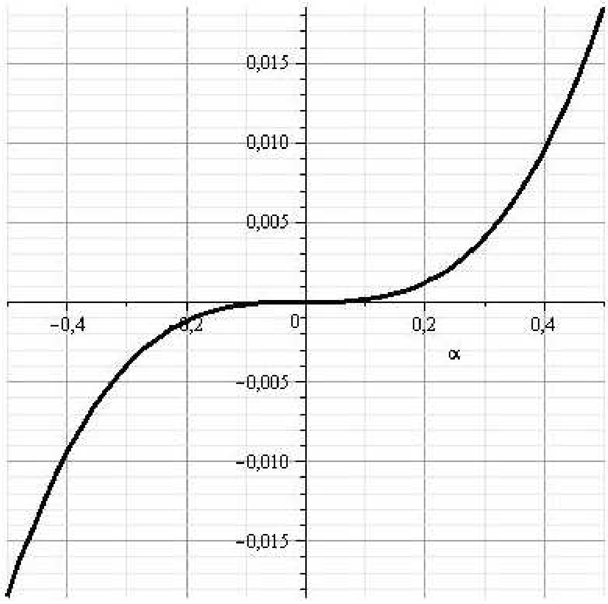


Figure 5. Bifurcation curve

From the above mentioned it follows that for values of $\alpha$ and $\beta$ over the bifurcation curve, i.e. when

$$\beta > \frac{4}{27}\alpha^3 \tag{19}$$

conditions for oscillations do not exist. In case of this combination of the parameters the mass will accomplish only one displacement with variable acceleration. The closer to the magnet is the mass, the higher is its velocity. This kind of motion and the functions $f$, $f_1$ and $f_2$, are shown in Figure 6. Vice versa, if $\alpha$ and $\beta$ are with values under the bifurcation curve, both oscillating or aperiodical one-way motion are possible. This kind of motion and the functions $f$, $f_1$ and $f_2$ are shown in Figure 7.

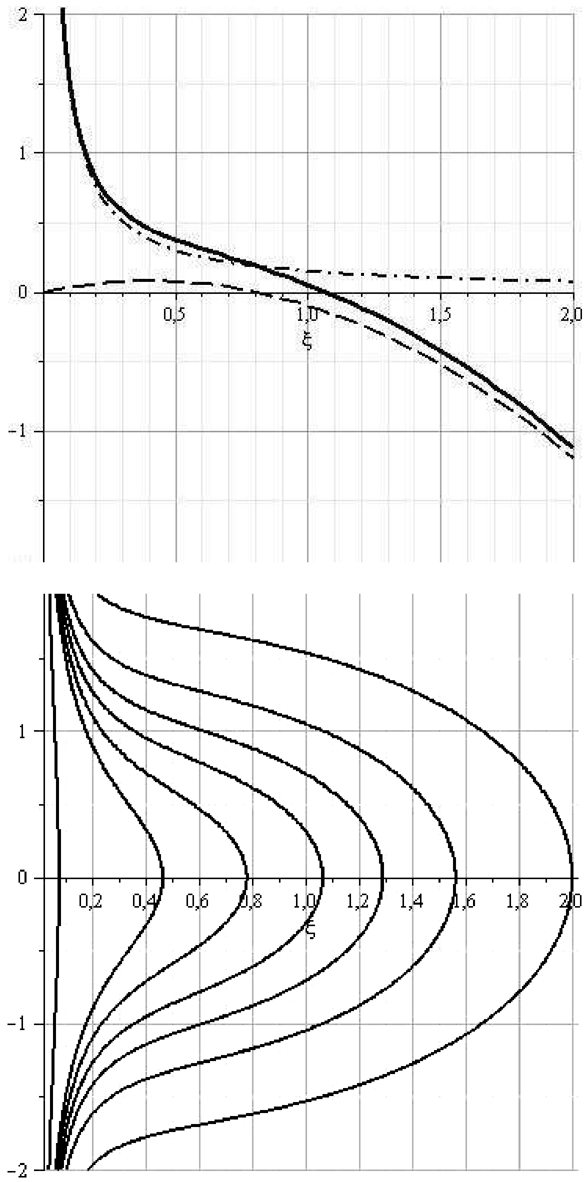


Figure 6. Typical phase portraits of the system with values of $\alpha$ and $\beta$ over the bifurcation curve

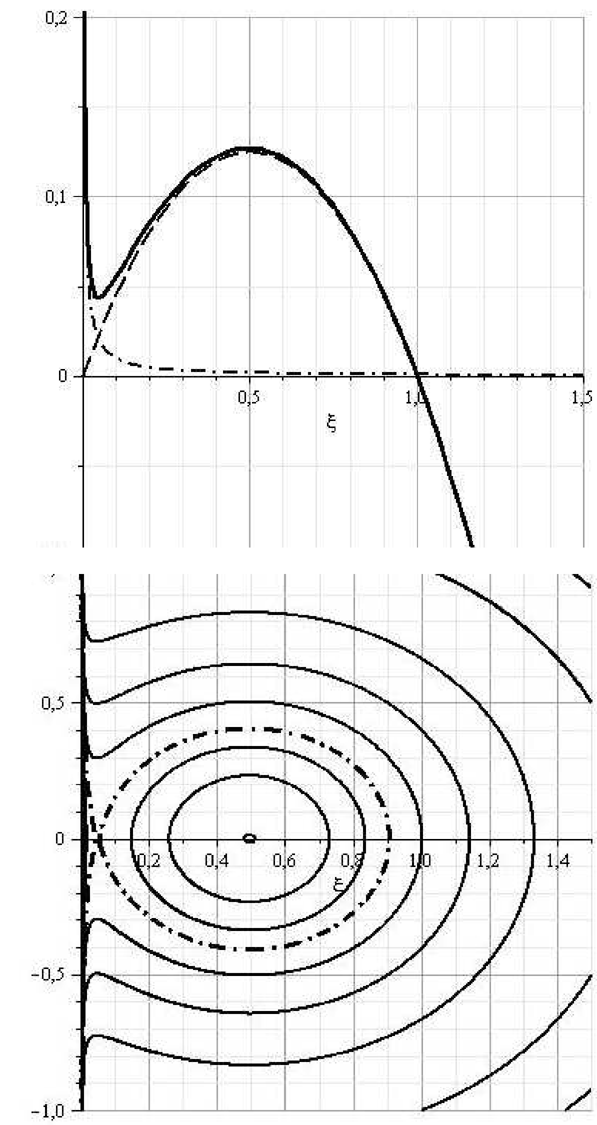

Figure 7. Typical phase portraits of the system with α and β under the bifurcation curve

## 5. Equilibrium point investigation

The equilibrium or singular points [5, 7, 8] of the oscillating system follow from the system of algebraic equations

$$\ddot{\xi}=0, \qquad \dot{\xi}=0. \tag{20}$$

The second equation shows that the singular points lie on the axis O$\xi$. From the first equation taking of account (8) it follows

$$-\xi+\alpha-\frac{\beta}{\xi^2}=0 \tag{21}$$

or for $\xi \neq 0$ it is yield

$$-\xi^3+\alpha\cdot\xi-\beta=0\,. \tag{22}$$

This cubic equation [9] has a discriminate

$$D=-\frac{1}{27}\beta\cdot\alpha^3+\frac{1}{4}\beta^2. \tag{23}$$

In order the Eq. (22) to possess three different roots it is necessary

$$D=-\frac{1}{27}\beta\cdot\alpha^3+\frac{1}{4}\beta^2<0\,, \tag{24}$$

from where the condition is yield

$$\beta<\frac{4}{27}\alpha^3. \tag{25}$$

This result confirms again the proof that the equilibrium points can exist only if the parameters α and β are chosen under the bifurcation curve. In this case there are three equilibrium points, but as it can be seen on Figure 7, only two of them are positive. This statement can be proved easily. The equilibrium point with the biggest positive coordinate is a stable center. The second positive equilibrium point is an unstable saddle. Through it the phase curve **S** passes, called separatrix. This separatrix splits the phase plane in areas with stable oscillations and areas with aperiodical single displacements with variable accelerations.

## 6. Dynamical synthesis of the oscillating system

The aim of the dynamical synthesis of the magneto-mechanical system is expressed here in finding of such combination of the parameters, which guarantees periodic oscillations.

Firstly, a magnet couple with imaginary magnet mass $\kappa$ = 0.6268×10$^{-6}$ Nm$^2$ and imaginary initial gap $\delta_0$ =1.2348×10$^{-3}$ m according to formula (1) is chosen. From the expressions (9) with a given natural frequency $k$ = 400 s$^{-1}$ it is found $\alpha = h_0 - g/k^2$ = 0.004172. A mass $m$ = 0.005 kg is chosen and the condition (24) is checked up, finding initially $\beta = \kappa / m\cdot k^2$ = 7.924×10$^{-10}$ and after substituting in $\beta > 4\alpha^3 / 27$ = 1.075×10$^{-8}$ – obtaining the necessary relation. The stiffness of the system is found by $c = m\cdot k^2$ = 800 N/m. The obtained parameters are sufficient for choosing the rest of the geometrical parameters and material properties.

The so chosen values do not guarantee oscillations with the prescribed mechanical frequency, because the system is nonlinear and in the considered case it is proved that the natural frequency is lower than the mechanical one. On bigger amplitudes because of isohornity of the system the frequency depends on the initial conditions.

The shape of the synthesized functions $f$, $f_1$, $f_2$ and phase portrait of the system is shown in Figure 8. From the figure it can be seen that the system has only one singular point corresponding to an equilibrium point of type focus. The next equilibrium point which is unstable gets in the zone $\xi < \delta_0$ and it could not be reached because of the stroke constraint. The same is relevant to the third equilibrium point, which is negative. The constraint $\xi \geq \delta_0$ changes the nature of motion of the system for the area inside the separatrix **S**, because here constraints for the periodical motions appear and in this way they turn in aperiodical ones.

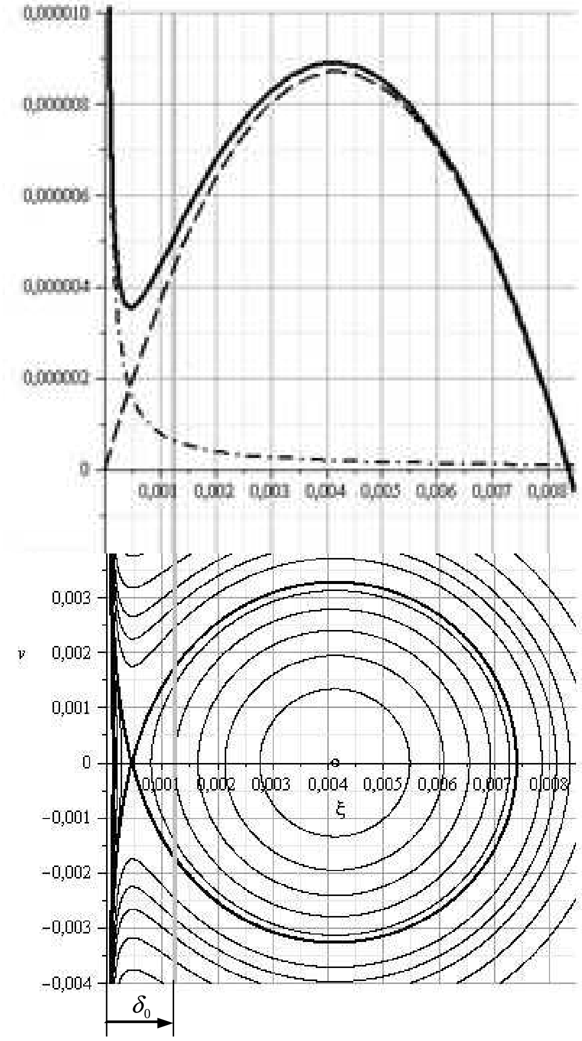


Figure 8. Phase portrait of the synthesized system

## 7. Conclusion

The qualitative analysis of the considered magneto-mechanical system allows the properties of the system to be discovered. This analysis is a unique mean of studying similar systems in the cases when the differential equation is nonlinear and cannot be solved exactly, or it leads to solutions which cannot be analyzed. On the base of the qualitative analysis it is possible a preliminarily dynamical synthesis of the real oscillating system to be achieved.

### References

1. Konstantinov M.C.: *Theory of mechanisms and machines*. Tehnika, Sofia, 1959, p. 493-501 (in Bulgarian)
2. Litvin, F.L.: *Design of Mechanisms and Instrument Elements*. Mashinostroene, Moscow, 1973 (in Russian)
3. Levitskii, N.I.: *Theory of mechanisms and machines*. Nauka, Moscow, 1979, p. 280-292 (in Russian)
4. Jordan, D.W., Smith, P.: *Nonlinear ordinary differential equations*. 3rd Ed. Oxford University Press Inc., New York, 2007, p. 2-126
5. Butenin, N.V., Nejmark, U.I., Fufaev, N.A.: *Introduction in nonlinear theory of oscillations*. Nauka, Moscow, 1987, p. 5-51 (in Russian)
6. Timoshenko, S., Woinovski-Krieger, S.: *Theory of Plates and Shells*. McGraw-Hill INC, New York, 1959, p. 63-76
7. Rothwell, E.J., Cloud, M.J.: *Electromagnetics*. CRC Press LLC, London, 2001, p. 15-31
8. Khalil, K.H.: *Nonlinear Equations*. 2nd Edition, Rmuee-HdI. Inc, London, ISBN 0-13-P8U24-8, 1996, p. 57-88
9. Bronshtein, I.N.,·Semendyayev, K.A., Musiol, G.,Muehlig, H.: *Handbook of Mathematics*. 5th Ed. Springer, ISBN 978-3-540-72121-5, 2007, p. 39-43

Received in February 2009