A NEWTON – EULER APPROACH TO THE FREELY SUSPENDED LOAD SWINGING PROBLEM

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Abstract. The present article is dedicated to the creation of a mathematical model for solving the problem of swinging of a freely suspended load, the pivot point of which moves along a known, spatial, time dependent or position dependent trajectory. The motion of the pendulum is described as rotations about two mutually perpendicular axes. Thus the Newton – Euler dynamic formulation is utilized and the equilibrium of a spherical pendulum under influence of a set of gravity and inertia forces and moments is considered. This approach leads to a system of two straightforward ordinary differential equations which could be solved numerically using any available solver. The solution is easy to programmes and could be used for dynamic investigation of the scope of different types of equipment.

Keywords: payload, swinging, dynamics, mathematical model

1. Introduction

The swinging of a freely suspended load occurs in many hoisting devices such as overhead cranes, jib-cranes, hydraulic automobile cranes, etc. In these cases the suspension point moves along a path in space which path is in general different for different kinds of equipment. This path could also be assumed as known in advance as it is assured by the driving system of the equipment, and combination of different working motions along certain degrees of freedom. Applying acceleration to the suspension point causes spatial swinging of the payload. Such motion produces additional inertia forces upon the construction at the pivot point (which could be quite substantial with big payloads and fast accelerations) and leads to increasing the working cycle of the installation due to the time elapsed in the process of stopping the oscillations.

The problem, as a question of present interest has been investigated by many authors from different points of view. Some publications (Jerman [1]) consider payload spatial motion for different type of cranes. Work [2] deals with modelling and study of a spherical pendulum response to different kinds of pivot point kinematics excitations, while some papers [3, 4, 5, 6] are devoted to the control of mechanical system motion and the generation of stabilizing control laws in order to decrease the payload swinging. [Mitrev et all, 2008] consider similar problems and use the Lagrange equations approach to solve the task.

The completed literature survey shows the considerable interest of many investigators towards the spatially swinging load problem. It could however be pointed out that in view of the multiple aspects of the problem additional discussion and research, finding new approaches to the solution of the general task are required.

The present article is devoted to the problem of creating a straightforward mathematical model of spatial oscillations for a freely suspended heavy load with a suspension point moving along a predefined path in space. Such a model will help solving several basic tasks:

- Evaluating the amplitude of the oscillating load for different working condition and different types of equipment;
- Evaluating the influence of additional inertia forces on the stability and structural integrity of the construction in many cases of deployment;
- Serving as basic research for designing a suitable trajectory function (control law) for efficient damping of the load movement, especially during the braking time and for stopping the motion.

2. Mathematical model – general considerations

The Newton – Euler approach is used for the solution of the problem as specified above. The graphical representation of the mechanical model is shown in Figure 1. The spatial swinging of the freely suspended payload can be decomposed into oscillations in two mutually perpendicular planes as shown in the figure. A basic local frame {1} is considered, its origin coincident with the pivot and the positive direction of axis Z_1 oriented upwards (opposite the direction of the gravity force). Two further frames of reference are defined, {2} and {3} in accordance with the Denavit – Hartenberg convention [7]. The origins of all three frames of references are coincident; the axis Z_2 is perpendicular on the first plane of oscillations while the axis Z_3 is perpendicular on the second plane; the axis X_2 is chosen normal to the plane of Z_2 and Z_3. Thus the deviation of the load in each plane is described by the independent variables θ_2 and θ_3 which also define the angles of rotation about Z_2 and Z_3 axes, respectively. In Figure 1 the oscillations in both planes and directions of Z axes are visualized by a massless cross part called conditionally "link2". The suspended load can be also referred to as "link3" for the sake of brevity. Frame {2} is considered rigidly attached to link2 and frame {3} rigidly attached to link3.

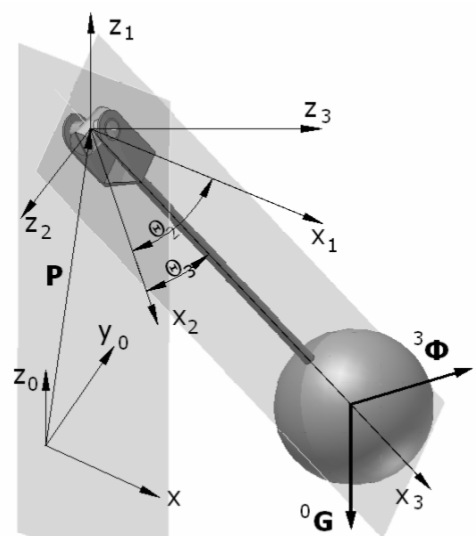

Figure 1. Mechanical model

Thus the rotational matrix R_2^1 describes the orientation of system {2} in respect to system {1}, while the rotational matrix R_3^2 describes the orientation of system {3} in respect to system {2} accordingly:

$$R_2^1 = \begin{bmatrix} cos\theta_2 & -sin\theta_2 & 0 \\ 0 & 0 & -1 \\ sin\theta_2 & cos\theta_2 & 0 \end{bmatrix}$$
$$R_3^2 = \begin{bmatrix} cos\theta_3 & -sin\theta_3 & 0 \\ 0 & 0 & 1 \\ -sin\theta_3 & -cos\theta_3 & 0 \end{bmatrix} \tag{1}$$

Further needed is the orientation of the third system relative to system {1} which is given by $R_3^1 = R_2^1 \cdot R_3^2$. As the columns of the rotational matrices represent projections of orthogonal unit vectors, for each i and j (i, j = 1…3) it can be written: $R_j^i = (R_i^j)^{-1} = (R_i^j)^T$.

The notations $\boldsymbol{\omega}_i$ and $\dot{\boldsymbol{\omega}}_i$ (i = 2,3) refer to the angular velocity and angular acceleration vectors of each link expressed in terms of its own frame. Considering the possible motion of each link as rotation about the Z axis of its own frame of reference system and using the propagation of angular velocity and acceleration between links it can written:

$$\dot{\boldsymbol{\theta}}_i = \begin{bmatrix} 0 & 0 & \dot{\theta}_i \end{bmatrix}^T, \ \ddot{\boldsymbol{\theta}}_i = \begin{bmatrix} 0 & 0 & \ddot{\theta}_i \end{bmatrix}^T$$
$$\boldsymbol{\omega}_2 = \begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix}^T = \dot{\boldsymbol{\theta}}_1 \tag{2}$$
$$\dot{\boldsymbol{\omega}}_2 = \begin{bmatrix} 0 & 0 & \ddot{\theta}_2 \end{bmatrix}^T = \dot{\boldsymbol{\theta}}_2$$

$$\boldsymbol{\omega}_3 = R_2^3 . \boldsymbol{\omega}_2 + \dot{\boldsymbol{\theta}}_3$$
$$\dot{\boldsymbol{\omega}}_3 = R_2^3 . \ddot{\boldsymbol{\theta}}_2 + \ddot{\boldsymbol{\theta}}_3 + C_{\varepsilon 3} \tag{3}$$
$$C_{\varepsilon 3} = R_2^3 . \boldsymbol{\omega}_2 \times \dot{\boldsymbol{\theta}}_3$$

where $\dot{\theta}_i$ and $\ddot{\theta}_i$ are the magnitudes of angular velocity and acceleration vectors along each degree of freedom.

Frame {1} moves in respect to the same grounded frame of reference {0} in such a way that the respective axes of the both systems stay parallel. The position of frame {1} (or the pivot point) relative to frame {0} is given by a positional vector $\mathbf{P}^0$ the coordinates of which are expressed in frame {0} (the same in {1} - $\mathbf{P}^0 = \mathbf{P}^1$) and are time (or position) dependent, or:

$$\mathbf{P}^1(t) = \begin{bmatrix} x_p(t) & y_p(t) & z_p(y) \end{bmatrix}^T,$$
$$\dot{\mathbf{P}}^1(t) = \begin{bmatrix} v_p(t) & v_p(t) & v_p(y) \end{bmatrix}^T \tag{4}$$
$$\ddot{\mathbf{P}}^1(t) = \begin{bmatrix} a_p(t) & a_p(t) & a_p(y) \end{bmatrix}^T$$

It is assumed that these functions are known, continuous, and smooth. Such a function depends on the type of the equipment used and could result from certain combined motions (for example accelerated rotation of the jib-cane while simultaneously elevating the jib, etc.). Knowing the motion law of the suspension point allows computing the acceleration of the gravity centre for link3:

$$\dot{\mathbf{v}}_{C3} = \dot{\boldsymbol{\omega}}_3 \times \mathbf{P}_{C3} + C_{VC3}$$
$$C_{VC3} = \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{P}_{C3}) + R_1^3 . \ddot{\mathbf{P}}^1 \tag{5}$$

Here $\mathbf{P}_{C3}$ is the gravity centre vector for link3 expressed in its own frame of reference.

3. Newton – Euler dynamic formulation

Next the apply Newton-Euler dynamic formulation is applied to the problem. Figure 1 shows the forces acting at the mass centre of link3: force $\mathbf{\Phi}$ causing the acceleration as well as weight $\mathbf{G}$ of the payload. The latter is best known in frame {0}, (respectively frame {1}) and could be simply described as $\mathbf{G}^1 = [0 \quad 0 \quad -m.g]^T$ where m is the mass of link3, and g - gravity acceleration.

Under the influence of all forces applied – gravity and inertia (additional forces such as friction, environment resistance, forces caused by wind etc. could also be added here), link3 is in equilibrium at any moment and the equation of equilibrium of the torque can be written by summing the torques about the origin of frame {3} and setting the sum equal to zero:

$$\begin{aligned} & -m.(\mathbf{P}_{C3} \times \dot{\mathbf{v}}_{C3}) - I.\dot{\boldsymbol{\omega}}_3 - \boldsymbol{\omega}_3 \times I.\boldsymbol{\omega}_3 + \\ & \quad + \mathbf{P}_{C3} \times R_1^3.\mathbf{G}^1 = 0 \end{aligned} \tag{6}$$

In the formula above $-I.\dot{\boldsymbol{\omega}}_3 - \boldsymbol{\omega}_3 \times I.\boldsymbol{\omega}_3$ represents Euler's expression for the moment acting on the body which moves with angular velocity $\boldsymbol{\omega}_3$ and angular acceleration $\dot{\boldsymbol{\omega}}_3$, I is the inertia tensor of the link written in a frame the origin of which is located at the centre of mass and axes parallel to those of frame {3}

Substituting (3) and (5) in (6) and also considering (2) it can be written:

$$\begin{aligned} & m.\mathbf{P}_{C3} \times [(R_2^3.\ddot{\boldsymbol{\theta}}_2 + \ddot{\boldsymbol{\theta}}_3) \times \mathbf{P}_{C3}] + \\ & \quad + I.(R_2^3.\ddot{\boldsymbol{\theta}}_2 + \ddot{\boldsymbol{\theta}}_3) = \\ & = \mathbf{P}_{C3} \times {}_1^3R.\mathbf{G}^1 - \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{P}_{C3}) - \\ & \quad - m.\mathbf{P}_{C3} \times C_{VC3} - I.C_{\varepsilon 3} \end{aligned} \tag{7}$$

In the equation above all terms containing accelerations $\ddot{\boldsymbol{\theta}}_i$ are grouped on the left side, while all other terms are moved to the right hand side. In the left hand side also the matrix operator of the cross product can be used, and subsequently the terms can regrouped. Thus the first second order ordinary differential equation is obtained:

$$\begin{aligned} & [(\tilde{P}_{C3}.(-\tilde{P}_{C3}) + I].R_2^3.\ddot{\boldsymbol{\theta}}_2 + \\ & \quad + [(\tilde{P}_{C3}.(-\tilde{P}_{C3}) + I].\ddot{\boldsymbol{\theta}}_3 = \\ & = \mathbf{P}_{C3} \times {}_1^3R.\mathbf{G}^1 - \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{P}_{C3}) - \\ & \quad - m.\mathbf{P}_{C3} \times C_{VC3} - I.C_{\varepsilon 3} \end{aligned} \tag{8}$$

where $\mathbf{P} \times = \tilde{P}$ represents the cross product involved by a matrix operator. $\tilde{P}$ is the skew-symmetric matrix associated to vector $\mathbf{P}$, *i.e.*, $\mathbf{P} \times \mathbf{E} = \tilde{P}.\mathbf{E}$ for any three-component vector $\mathbf{E}$.

$$\mathbf{P}_{C3} \times = \tilde{P}_{C3} = \begin{bmatrix} 0 & -P_{C3x} & P_{C3y} \\ P_{C3x} & 0 & -P_{C3z} \\ -P_{C3y} & P_{C3z} & 0 \end{bmatrix} \tag{9}$$

Considering the equilibrium of virtual link2, there will be no additional inertia, gravity forces or torques to be added. Expressing the equation (8) in terms of frame (2) it can be simply written:

$$\begin{aligned} & R_3^2.[(\tilde{P}_{C3}.(-\tilde{P}_{C3}) + I].R_2^3.\ddot{\boldsymbol{\theta}}_2 + \\ & \quad + R_3^2.[(\tilde{P}_{C3}.(-\tilde{P}_{C3}) + I].\ddot{\boldsymbol{\theta}}_3 = \\ & = R_3^2.(\mathbf{P}_{C3} \times {}_1^3R.\mathbf{G}^1 - \boldsymbol{\omega}_3 \times (\boldsymbol{\omega}_3 \times \mathbf{P}_{C3}) - \\ & \quad - m.\mathbf{P}_{C3} \times C_{VC3} - I.C_{\varepsilon 3}) \end{aligned} \tag{10}$$

To numerically solve the problem the ordinary differential equations can be integrated together:

$$\begin{aligned} A_{32}.\ddot{\boldsymbol{\theta}}_2 + A_{33}.\ddot{\boldsymbol{\theta}}_3 = B_3 \\ A_{22}.\ddot{\boldsymbol{\theta}}_2 + A_{23}.\ddot{\boldsymbol{\theta}}_3 = B_2 \end{aligned} \tag{11}$$

Note that the above coefficients A_{ij} have the form of 3×3 matrixes, while B_i represents 1×3 a column vector. For the purpose of equations integration any numerical integration procedure is suited, such as the ODE45 solver of the MATLAB package for example. Having in mind that the oscillations occur only along the Z axis of the respective frame of reference, for every iteration only the lower left term of the A_{ij} matrices and the third term of the B_i vectors can be considered, leading to a system of linear algebraic equations for $\ddot{\theta}_2$ and $\ddot{\theta}_3$.

4. Results and conclusion

The set of ordinary differential equations is numerically integrated for the repositioning of the suspension point along a spatial trajectory shown in Figure 2, (curve 1; Curve 2 shows the projection of the path on XY plane). The time span for moving the point between the initial and the endpoint is set to 6 seconds. Combining oscillations of the load about both degrees of freedom, computed according to equation (11) allows determining the position of the mass centre along the path which is presented in Figure 2 (curve 3).

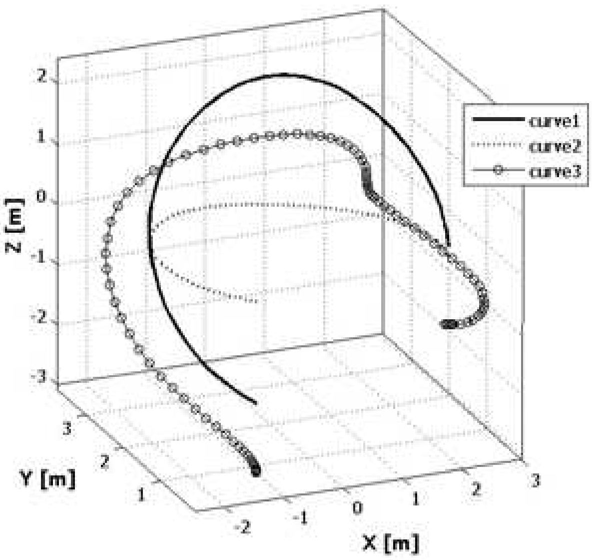

Figure 2. Trajectories:
curve 1 – trajectory of the pivot point;
curve 2 – projection on the XY plane;
curve 3 – trajectory of the load

Equation (10) provides a direct approach to computing the accelerations along each degree of freedom, which in turn allows finding the acceleration vector for the mass centre of the pendulum. Thus the inertia forces can be obtained, acting at the mass centre and (by adding the weight of the load) at the pivot point, respectively (Figure 3).

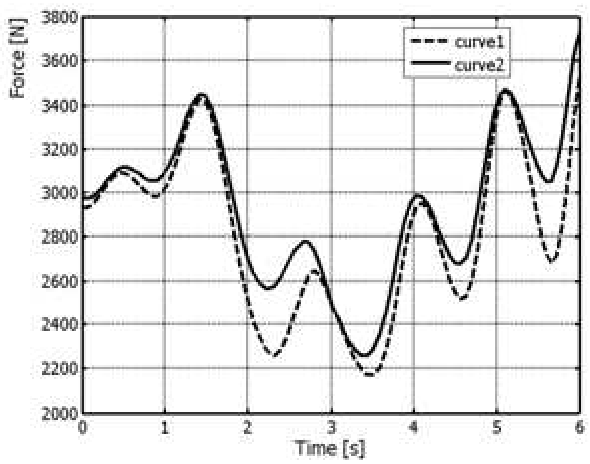

Figure 3. Force acting at the pivot point:
curve1 – Z component;
curve2 – full force

The proposed approach leads to a set of two non-linear ordinary differential equations allowing the determination of the oscillations of a freely suspended load. The derived equations are easily programmed and can be numerically integrated by any available solver. The investigations show that with the intensification of the load transferring processes (using higher accelerations and combining several working motions) a considerable swinging of the suspended load can be expected, and considerable forces acting on the construction which are to be taken into account for the structural integrity or fatigue analysis. The proposed approach could be used for dynamic investigation of the scope of different types of equipment.

References

1. Jerman, B.: *An enhanced mathematical model for investigating the dynamic loading of a slewing crane*. Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science Vol. 220, Issue 4, 2006, p. 421-433, ISSN 0954-4062
2. Leung, A.Y.T, Kuang, J.L.: *On the Chaotic Dynamics of a Spherical Pendulum with a Harmonically Vibrating Suspension*. Nonlinear Dynamics, Volume 43, Issue 3, 2006, p. 213-238, ISSN 0924-090X, Springer Netherlands
3. Eihab, A.-R., Nayfeh, A.H.: *Pendulation Reduction in Boom Cranes Using Cable Length Manipulation*. Nonlinear Dynamics, Vol. 27, Issue 3, 2002, p. 255-269, ISSN 0924-090X, Springer Netherlands
4. Masoud, Z.: *A Control System for the Reduction of Cargo Pendulation of Ship-Mounted Cranes*. Ph.D. dissertation, December 4, 2000, Blacksburg, Virginia, USA
4. Pauluk, M., Korytowski, A., Turnau, A., Szymkat, M.: *Time optimal control of 3D crane*. Proceedings of 7th IEEE International Conference MMAR 2001, 28-31 Aug. 2001, p. 927-932, Miedzyzdroje, Poland
4. Bockstedte, A., Kreuzer, E.: *Hoisting manipulation by modal coupling control for underactuated cranes*. Solid mechanics and its application. Vol. 130, p. 121-130, 2005, ISSN 0925-0042, Kluwer Academic Publishers, Netherlands
5. Craig, J.: *Introduction to robotics. Mechanics and control*. Addison-Wesley publishing company, ISBN 0-201-10326-5, California, 1989

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