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ROLLING DISK DYNAMICS

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Abstract. The present paper aims to establish the differential equations of the motion for a homogeneous disk, which is rolling upon a rough horizontal plane. The results will prove useful in a future research, when the motion of a monowheel vehicle will be studied.

Keywords: rolling, disk, motion, monowheel

1. Introduction

The first monowheel vehicle was built by Rousseau, a craftsman from Marseilles, and dates back to 1869 (figure 1). As there is no steering mechanism, the rider must have a good sense of balance. This is probably the explanation of the delay in the extension of this simple and economical vehicle.

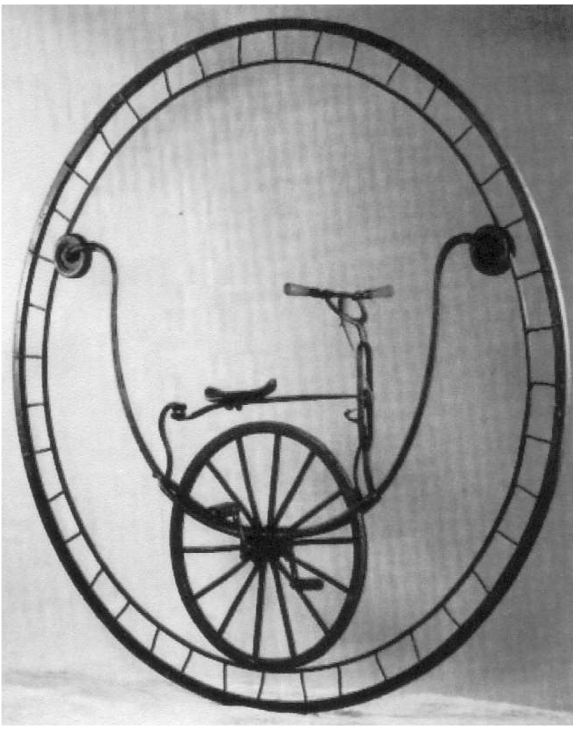

Figure 1. Rousseau's monowheel

Lately, thanks to the improvement in engine science and in material technology developments, the monowheel vehicle became both a useful and a fun vehicle. One of many recent models of hand-built monowheel vehicles is the one made by Kerry Mclean in the 2001 (figure 2). The presented model has a 5 HP engine and costs about $8500 [1].

Figure 2. Kerry Mclean's monowheel

2. Rolling disk motion

2.1. Generalities

The rolling disk motion is a somehow difficult problem to put in equation, because of non-holonomous character of the disk link with the support plane (the road plane). The disk has three degrees of freedom, but in order to find the motion equations, we need five coordinates. The differentials of these five coordinates, which represent the virtual displacements, should verify the two Pfaff's equations of the link (rolling conditions), and then Lagrange's equations will involve two amplifiers.

2.2. Reference frames

First, we shall take a fixed reference frame $O_1x_1y_1z_1$, having the origin in the road plane, and the O_1z_1 axis pointing vertically upward (Figure 3). The second reference frame is $Cxyz$ that is bound to the disk and oriented to its principal central inertia directions. The disk orientation is established by Euler's angles θ, φ, ψ.The third reference frame will be $O'x'y'z'$, which is always parallel to $Cxyz$ -frame, but having a fixed origin.

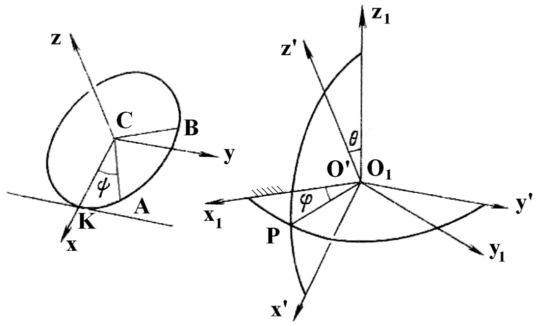

Figure 3. Reference frames

The coordinates of the centre of mass C will be $\xi, \eta, a\sin\varphi$, where *a* – the disk radius.

2.3. Holonomous and non-holonomous systems

In the most general case, a link equation can be written in a Pfaff form

$$a\,dx + b\,dy + c\,dz + p\,dt = 0, \tag{1}$$

where *a, b, c, p* are given functions belonging to the class C_1 of the variables *x, y, z, t*. Let the displacements corresponding to equation (1) be some possible displacements. In the same way, the virtual displacements verify the equation

$$a\,\delta x + b\,\delta y + c\,\delta z = 0. \tag{2}$$

As a rule, the possible displacements are different from the virtual ones, but, if the function *p* is always equal to zero, these kind of displacements are identical. In this case, the system is denominated *catastatical.* So, a catastatical system is characterized by *i*) the identity between the virtual displacements and the possible ones, and *ii*) that speed $\{\dot{x}, \dot{y}, \dot{z}\} = 0$ is a possible speed.

If the Pfaff's equation (1) of the link can be integrated (after its amplification with a corresponding integrating factor), then we shall say the system is *holonomous.* In this case, the link equation can be written in the final form

$$\gamma(x, y, z, t) = 0 \tag{3}$$

Of course, if the equation (1) cannot be integrated, the system is called *non-holonomous.*

In order to answer to the question of the basic difference between a holonomous system and a non-holonomous one, it is enough to examine the catastatical systems, taking into account only the simple cases in which the coefficients *a, b, c* do not depend of time.

If the Pfaff form $a\,dx + b\,dy + c\,dz$ does admit an integrating factor, then the system is holonomous and the link equation can be written as

$$\Gamma(x, y, z) = const. \tag{4}$$

Therefore, it results that, from every given point (let us say, the coordinates origin) a two-parametric point set may be sufficient, and namely the points of the surface

$$\Gamma(x, y, z) = \Gamma(0, 0, 0). \tag{5}$$

On the other hand, if the system is non-holonomous, a three-parametric point set may be sufficient. For example, let the link equation be given in the form

$$dy - z\,dx = 0, \tag{6}$$

which, evidently, does not admit an integrating factor. In this case, we may find a solution corresponding to the condition (6), and leading from the origin to an arbitrary point x_2, y_2, z_2. In order to demonstrate this, let us examine the solution

$$y = f(x), \quad z = f'(x), \tag{7}$$

with $f(x) \in C_2$. The link equation (6) is, evidently, satisfied, and what remains, is to chose the *f(x)* function form so that it verifies the conditions

$$f(0) = 0,\ f'(0) = 0,\ f(x_2) = y_2,\ f'(x_2) = z_2. \tag{8}$$

2.4. Virtual displacements

Along a virtual displacement of the disk, the differentials $d\xi, d\eta, d\theta, d\varphi, d\psi$ are bound by two relationships (rolling conditions), namely [3]

$$d\xi\cos\varphi + d\eta\,\sin\varphi - a\,d\theta\sin\theta = 0, \tag{9}$$

$$-\,d\xi\sin\varphi + d\eta\cos\theta + a\,d\psi + a\,d\varphi\cos\theta = 0 \tag{10}$$

In this problem, of course, the virtual displacements and the possible displacements are coincidental. In order to find the proper form of the equations (2) and (3), we shall start from the condition that the point *K* of the disk has to be instantaneously at rest. So, the velocity of the centre *C* of the disk will have the components

$0, -a\omega_z, a\omega_y$ or $0, -a(\dot{\psi} + \dot{\varphi}\cos\theta), a\dot{\theta}.$

Now, the components of the same velocity vector on the directions O_1P and O_1y' will be equal respectively to $a\dot\theta\sin\theta,\ -a(\dot\psi+\dot\varphi\cos\theta)$.

Equalizing these expressions with $\dot\xi\cos\varphi+\dot\eta\sin\varphi,\ -\dot\xi\sin\varphi+\dot\eta\cos\varphi$ respectively, we obtain the rolling conditions (9) and (10).

3. Motion differential equations

3.1. Lagrange function

The Lagrange function is defined as the difference between the kinetic energy and the potential energy of the system, namely [2]

$$L=\frac{1}{2}M\left(\dot\xi^2+\dot\eta^2+a^2\cos^2\theta\,\dot\theta^2\right)+ \frac{1}{2}J_x\left(\dot\theta^2+\dot\varphi^2\sin^2\theta\right)+ \frac{1}{2}J_z(\dot\psi+\dot\varphi\cos\theta)^2-Mga\sin\theta, \tag{11}$$

where M – mass of the disk, $J_x=J_y, J_z$ - its axial moments of inertia.

3.2. Lagrange equations

The equations of the rolling motion of the disk will be deducted with the help of the Lagrange function (11) and rolling conditions (9) and (10). They are as follows:

$$M\ddot\xi=\lambda\cos\varphi-\mu\sin\varphi \tag{12}$$

$$M\ddot\eta=\lambda\sin\varphi+\mu\cos\varphi \tag{13}$$

$$\frac{d}{dt}\left(Ma^2\cos^2\theta\dot\theta+J_x\dot\theta\right)=-Ma^2\cos\theta\sin\theta\dot\theta^2+ +J_x\cos\theta\sin\theta\dot\varphi^2-J_z\omega_z\dot\varphi\sin\theta-Mga\cos\theta- -\lambda a\sin\theta, \tag{14}$$

$$\frac{d}{dt}\left(J_x\sin^2\theta\dot\varphi+J_z\omega_z\cos\theta\right)=\mu a\cos\theta, \tag{15}$$

$$\frac{d}{dt}(J_z\omega_z)=\mu a. \tag{16}$$

Here, we have denoted by ω_z, the quantity $\dot\psi+\dot\varphi\cos\theta$. Altogether, we have seven equations: five equations (12)-(16), and the rolling conditions [3, 4]:

$$\dot\xi\cos\varphi+\dot\eta\sin\varphi=a\sin\theta\dot\theta, \tag{17}$$

$$-\dot\xi\sin\varphi+\dot\eta\cos\varphi=-a\omega_z. \tag{18}$$

The physical sense of the quantities λ,μ is obvious: from the equations (12) and (13) we get

$$M\left(\ddot\xi\cos\varphi+\ddot\eta\sin\varphi\right)=\lambda, \tag{19}$$

$$M\left(-\ddot\xi\sin\varphi+\ddot\eta\cos\varphi\right)=\mu, \tag{20}$$

in this respect, λ,μ *represent the components of the reaction force in the contact point K*, on the directions O_1P and O_1y'.

Now, it is easy to express λ,μ in terms of θ,φ,ψ and their derivatives; in this sense, from (17) and (18) we have

$$\ddot\xi\cos\varphi+\ddot\eta\sin\varphi+\left(-\dot\xi\sin\varphi+\dot\eta\cos\varphi\right)\dot\varphi= =a\left(\sin\theta\ddot\theta+\cos\theta\dot\theta^2\right), \tag{21}$$

$$-\ddot\xi\sin\varphi+\ddot\eta\cos\varphi-\left(\dot\xi\cos\varphi+\dot\eta\sin\varphi\right)\dot\varphi= =-a\dot\omega_z. \tag{22}$$

Considering the equations (17) – (20), we get

$$\lambda=Ma\left(\sin\theta\ddot\theta+\cos\theta\dot\theta^2+\omega_z\dot\varphi\right), \tag{23}$$

$$\mu=Ma\left(\sin\theta\dot\theta\dot\varphi-\dot\omega_z\right). \tag{24}$$

So, above, we have the expressions of the contact reaction components on the O_1P and O_1y' directions.

Now, replacing these expressions of λ and μ in the equations (14) and (16), and eliminating μ between (15) and (16), we find

$$\left(J_x+Ma^2\right)\ddot\theta=J_x\dot\varphi^2\cos\theta\sin\theta- -\left(J_z+Ma^2\right)\omega_z\dot\varphi\sin\theta-Mga\cos\theta, \tag{25}$$

$$\left(J_z+Ma^2\right)\dot\omega_z=Ma^2\dot\theta\dot\varphi\sin\theta, \tag{26}$$

$$\frac{d}{dt}\left(J_x\dot\varphi\sin^2\theta\right)=J_z\omega_z\dot\theta\sin\theta. \tag{27}$$

These three equations contain three unknowns: θ,φ,ω_z.

Now, we can constitute two differential equations which will express ω_z and $\dot\varphi$ as functions of θ. Denoting $\cos\theta$ by *p*, we can rewrite the equations (26) and (27) as

$$(2k+1)\frac{d\omega_z}{dp}=-\dot\varphi, \tag{28}$$

$$\frac{d}{dp}\left\{\left(1-p^2\right)\dot\varphi\right\}=-2\omega_z. \tag{29}$$

Here, we have denoted $J_x=kMa^2$ and $J_z=2kMa^2$. Eliminating $\dot\varphi$, we obtain

$$\frac{d}{dt}\left\{\left(1-p^2\right)\frac{d\omega_z}{dp}\right\}-\frac{2}{2k+1}\omega_z=0. \tag{30}$$

This differential Legendre equation determines ω_z in function of *p*. The value of the coefficient $\frac{2}{2k+1}$ is equal to 1 for a rim (considered a circular material line), and to $\frac{4}{3}$ for a disk. Eliminating ω_z from (28) and (29), we obtain

$$\left(1-p^2\right)\frac{d^2\zeta}{dp^2}=\frac{2}{2k+1}\zeta\,, \tag{31}$$

where ζ is equal to $\left(1-p^2\right)\dot{\varphi}$.

Finally, the differential equation (31) determines ζ, and $\dot{\varphi}$ consequently, as functions of p.

3.3. Steady-state turning

From the equation (25), we can obtain the conditions for the steady-state turning of the disk. That means the *regulated precession*, namely the motion in which the disk makes an angle α with the horizontal plane, and its centre describes a circle having the radius b, with the speed equal to $b\omega$.

Of course, such a case may be considered only if any friction is neglected (ideal case), or the disk is driven by an engine. The latter case is exactly the case of a monowheel vehicle.

In this steady-state turning, the value $\dot{\varphi}$ is equal to ω; here, the value ω is determined by the equation

$$\{(2k+1)b+ka\cos\alpha\}\omega^2=g\,ctg\alpha, \tag{32}$$

which can be easily deduced using the equality $-a\omega_z=b\omega$.

The presented calculi illustrate the way in which Lagrange equations can solve such a problem of non-holonomous systems. As one can see, this is entirely possible.

Starting from these results, the authors decided to continue the study of the monowheel motion, making the necessary adaptations for this. First, the principal difference is that monowheel rolling part is not a disk, but only a circular rim and tyre. Further, the inner body of a monowheel does not participate to all three rotations that the rim can make simultaneously; in fact, this inner frame has only a relative rotation against the outer rim, in the plane of the last one. More precisely, its angular speed about the axis normal to plane of the wheel (*Cy*- axis, in Figure 3) is independent of the wheel angular speed.

This study makes the object of another work presented in the same conference.

4. Conclusions

The monowheel vehicle is a system of bodies designed to transport one person on road, as well as off road. It differs from a motorcycle by the inside placement of the rider, as well as the great radius of the rolling wheel. This great diameter permits a good passing capacity over the off road irregularities and a small fuel consumption on the good road surfaces.

Its small dimensions in breadth permits a good circulation on the agglomerated streets of modern towns, as well as an easy parking.

Although at its beginning, the vehicle speed was very small, because the rider force drove them, after introducing gas engine motors the difficulties of maintaining the direction and stability were eliminated. Besides, they are very good means to take a ride during holidays.

The present paper represents the introductory part in the study on the serious problem of the motion of such a vehicle. The difficulty is generated by its non-holonomous links with the road surface, as we already pointed out.

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